Sequences and Series

A sequence is an array of elements seperated by commas Ex 1,4,7,10,13,........
The elements are called the terms of the sequence ie 1 is the first term U_1, 4 is the second term U_2 etc
General Term (Un) : This is an expression in n which describes every term of a particular sequence, the general term of the sequence 1,4,7,10 is Un = 3n-2 . A sequence can be describes in several ways
.(1) A sequence can be described using just the Un . A sequence Un = 4n+ 3 find the first three terms U_1=7,U_2= 11U_3 = 15 , Find U_{n+1}= 4(n+1) +3 = 4n+7 ..
(2) A sequence can be described recursively (a difference formula connecting up consecative terms of a sequence)
Example If $U_1 = 1/5, \ldots U_n = \frac{5}{2}U_{n-1} + \frac{3}{2}$ Find
$U_2, U_3 . U_2 = \frac{5}{2}(\frac{1}{5}) + \frac{3}{2} = 2, .. U_3 = \frac{5}{2}(2) + \frac{3}{2} = 6\frac{1}{2}$
The Syllabus requires you to be able to find the Sum of the first n terms of all of the following types of Sequences :
(1) Arithmetic $Un = a + (n-1)d$ (2) Geometric Un = ar^{n-1}(3) AP/GP $Un = a_n x^n$. (4) Telescopic type $Un = \dfrac{1}{(n)(n+1)}$

You will also be required to find the following limits $\lim_{n\to\infty} \dfrac{n}{n+1}, \ldots \lim_{n\to\infty} r^n \ldots |r| \langle 1$

Series
A series is the <u>sum of a sequence</u>, we use the symbol $\sum$ which means "the sum of " to describe a series .
$\sum_1^n n^2$ this is the series whose Un is n^2 from n = 1 to n = n . ie the sum of the first n terms of the series S_n
this is called the Sn of the Series $\sum_1^n Un = S_n$ and the sum to infinity of a series is
$\sum_1^\alpha Un = \lim_{n\to\alpha} Sn$ You will be required to find the Sn and the sum to infinity of all of the sequences described below .
The syllabus states that you can only be asked to consider Series whose Sn can be found and you will only be asked to find limits for sequences where Un is given explicitly .

<u>Must Know</u>

Arithmetic Series Un = a + (n-1)d , Sn = n/2 {2a+ (n-1)d}
Geometric Series Un = $ar^{n-1} . S_n = \dfrac{a(1-r^n)}{(1-r)} .. |r| < 1 .. S_\infty = \dfrac{a}{1-r}$

$$\sum_1^n n = \tfrac{n}{2}(n+1) .. \sum_1^n n^2 = \tfrac{n}{6}(2n+1)(n+1) .. \sum_1^n n^3 = \{\tfrac{n}{2}(n+1)\}^2$$

Telescopic Series $\sum_1^n \dfrac{1}{n(n+1)} = \dfrac{a}{n} + \dfrac{b}{n+1}$ (Partial fractions)

Arithmetic Series :

Key points : In this type of Series a constant is added to each term to give the next term . This constant is called the common difference (symbol d) , in all Arithmetic Series

$U_3 - U_2 = U_2 - U_1 \Rightarrow U_1 + U_3 = 2U_2 \Rightarrow U_2 = \frac{U_1 + U_3}{2}$ **.If the first term is(a) then an Arithmetic Series can be written as a + a+d +a+2d+a+3d +**

The General term is $U_n = a + (n-1)d$ **,The Sum of the first n terms is** $S_n = \frac{n}{2}\{2a + (n-1)d\}$

Example : Given the Arithmetic Series 1+5+9+13 +, find a,d,Un,Sn, .

a = 1, (5 - 1) == 4 = d , Un = 1 + (n-1)4 , Un = 4n - 3 . Sn = n/2{2(1) + (n-1)4} = n/2{4n - 2} .

Example 2: the first three terms of an Arithmetic Sequence are 6,-9,x find x .

-9-6 = x - - 9, x = 24 .

Example 3 :In an Arithmetic Sequence The sum of three consecative terms is -9 and there product is 48 . Method write the three terms as a-d, a, a+d

$a - d + a + a + d = -9 \Rightarrow 3a = -9 \Rightarrow a = -3 \ldots (a-d)(a)(a+d) \Rightarrow a(a^2 - d^2) = 48 \Rightarrow -3(9 - d^2) = 48$ **This gives**

$\Rightarrow 9 - d^2 = -16 \Rightarrow -d^2 = -25 \Rightarrow d = \pm 5.$

us the following sets of three terms -8.-3.2 .2,-3,8 .

Example 4 : In an Arithmetic Sequence

$$U_3 = 22, \ldots U_7 == 46 \Rightarrow \frac{a + 2d = 22}{a + 5d = 46} \Rightarrow -3d = -24 \Rightarrow d = 8, a = 6.$$

Example 5 : Sn of an Arithmetic Sequence is $3n^2 + 2n$ **Find** U_1, U_2, d, U_n

$S_1 = U_1 = 3(1)^2 + 2(1) = 5, S_2 = 3(2)^2 + 2(2) = 16 = U_2 + U_1 \Rightarrow S_2 - S_1 = 11 = U_2 \Rightarrow d = 6$

$U_n = 5 + (n-1)6 = 6n - 1.$

Geometric Series :

In this type of series each termis multiplied by a constant to give the next term . This constant is called (r) the common ratio .

In all geometric series $\frac{U_2}{U_1} = \frac{U_3}{U_2} = r. = \Rightarrow (U_2)^2 = U_1 U_3 \Rightarrow U_2 = \sqrt{U_1 U_3}$ **consecative terms of a geometric series can be written as a + ar + ar^2 +.... or a/r + a + ar + ...**

The general term is $Un = ar^{n-1}$

The Sum of the first n terms is $Sn = \frac{a(1 - r^n)}{1 - r} .. |r| < 1 \ldots Sn = \frac{a(r^n - 1)}{r - 1} .. r > 1.$

The sum to infinity of a Geometric Series exists only if $|r| < 1, .. S_\infty = \frac{a}{1 - r}$

Example 1:Find the Sum of the first 12 terms of the sequence

$$2, 2 \times 3, 2 \times 3^2, 2 \times 3^3, a = 2, r = 3, S_{12} = \frac{2(3^{12} - 1)}{3 - 1} = 3^{12} - 1.$$

Example 2:Find the Sum to infinity of the Series

$$1 + \frac{1}{1+x} + \frac{1}{(1+x)^2} \ldots (x > 0), a = 1, r = \frac{1}{1+x} \Rightarrow S_\infty = \frac{1}{1 - \frac{1}{1+x}} = \frac{1+x}{x}$$

Example 3 : The Sum of three consecative terms of a Geometric Series is 70 their product is 64000/27 Find two possible such series .

Write the terms as a/r,a, ar ,

$$\frac{a}{r}.a.ar = \frac{64000}{27} \Rightarrow a^3 = \frac{64000}{27} \Rightarrow a = \frac{40}{3}.$$

$$\frac{40/3}{r} + \frac{40}{3} + \frac{40}{3}r = 70 \Rightarrow 40 + 40r + 40r^2 = 70r \Rightarrow 40r^2 - 170r + 40 = 0 \Rightarrow 4r^2 - 17r + 4 = 0$$

$$(4r-1)(r-4) = 0 \Rightarrow r = 1/4, r = 4,..Series, 10/3, 40/3, 160/3 : or, 160/3, 40/3, 10/3$$

Example 4: Show that the Un of the Sequence 5,55,55,5555, can be written as the Sn of another Sequence . If we rewrite the sequence as follows 5, 5 + 5(10), 5+5(10)+5(10)^2, 5,5(10)+5(10)^2+5(10)^3 + ...

This sequence is Geometric where a = 5 , r = 10 and each term of the original sequence is the sum of the corresponding number of terms of the new sequence

$$U_1 = S_1, ..U_2 = S_2 \Rightarrow U_n = S_n = \frac{5(10^n - 1)}{10 - 1} = \frac{5}{9}(10^n - 1)$$

Arithmetico-Geometric Series "APGP" .

In this type of Series each term of an Arithmetic series is multiplied by the corresponding term of a Geometric series .

Ex ; the Series $1 + 3x + 5x^2 + 7x^3 + 9x^4 ....(2n-1)x^{n-1}$ **is an APGP**

The Ap is1+3+5+7+9+.......,(2n-1)

The GP is 1,+x+x^2+x^3+x^4 +...... .x^{n-1}

To find Sn of an APGP : (1) Write out the Series including Un .ie Sn =

(2) Multiply both sides by the common ratio of the GP . (3) Take line (2) from line (1) .

(4) You will be left with a "first term" a Gp and an "end Bit ". (5)Find the Sum of the Gp then divide your result by (1-r).

Example 1 : Find Sn of the series $1 + 3x + 5x^2 + 7x^3 + 9x^4 ....+(2n-1)x^{n-1}$

$$(1) Sn = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + ...+(2n-1)x^{n-1}.$$

$$(2) xSn = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 ....(2n-1)x^n$$

$$(1)-(2) \Rightarrow Sn(1-x) = 1 + \{2x + 2x^2 + 2x^3 + 2x^4 ....+2x^{n-1}\} - (2n-1)x^n$$

$$(4) Sn(1-x) = 1 + \frac{2x(1-x^{n-1})}{1-x} - (2n-1)x^n$$

$$(5) Sn = \frac{1}{1-x} + \frac{2x(1-x^{n-1})}{(1-x)^2} - \frac{(2n-1)x^n}{1-x}$$

The Gp has a = 2x, r = x and it has (n-1) terms .We find S(n-1) using the formula for Sn . 1 is not included in the GP and the end bit is (2n-1)x^{n-1} .

Telescopic Series (Series involving fractions) : These are Series of the form

$$\sum_1^n \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} .... \frac{1}{n(n+1)}.$$

$$\sum_1^n \frac{1}{(2n-1)(2n+1)} = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + ... \frac{1}{(2n-1)(2n+3)}.$$

To find Sn of this type of Series (1) Write Un as a pair of partial fractions

(2)Rewrite the series using the "new Un" (3) Simplify to find Sn

Find the sum of the following Series

$$\sum_{1}^{n}\frac{1}{n(n+1)},\ldots\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}=\frac{A(n+1)+Bn}{n(n+1)}\Rightarrow$$
$$A(n+1)+Bn=1, n=0\Rightarrow A=1, n=-1\Rightarrow B=-1\Rightarrow$$
$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$
$$U_1=\frac{1}{1}-\frac{1}{2}$$
$$U_2=\frac{1}{2}-\frac{1}{3}\ldots\ldots\ldots\ldots\ldots Sn=1-\frac{1}{n+1}$$
$$U_3=\frac{1}{3}-\frac{1}{4}$$
$$U_4=\frac{1}{4}-\frac{1}{5}$$
.

.

$$U_n=\frac{1}{n}-\frac{1}{n+1}$$

Find The Sum of the Series :

$$\sum_{1}^{n}\frac{1}{(2n-1)(2n+1)},\ldots\frac{1}{(2n-1)(2n+1)}=\frac{A}{(2n-1)}+\frac{B}{(2n+1)}\Rightarrow$$
$$1=A(2n+1)+B(2n-1),$$
$$n=\frac{1}{2,}\Rightarrow A=\frac{1}{2}, n=\frac{-1}{2}\Rightarrow B=\frac{-1}{2}$$
$$\Rightarrow U_n=\frac{1}{2}\{\frac{1}{(2n-1)}-\frac{1}{(2n+1)}\}$$
$$U_1=\frac{1}{2}\{\frac{1}{1}-\frac{1}{3}\}$$
$$U_2=\frac{1}{2}\{\frac{1}{3}-\frac{1}{5}\}\ldots\ldots\ldots$$
$$U_3=\frac{1}{2}\{\frac{1}{5}-\frac{1}{7}\}$$
.

.

$$U_n=\frac{1}{2}\{\frac{1}{(2n-1)}-\frac{1}{(2n+1)})$$
$$\Rightarrow S_n=\tfrac{1}{2}\{1-\frac{1}{(2n+1)}\}$$

The $\sum n, \sum n^2, \sum n^3$ these Series are (1) the Sum of the first n Natural numbers (2) The sum of the squares of the first n Natural numbers ,(3) the sum of the cubes of the first n natural numners .

$$\sum_{1}^{n} n=1+2+3+4+5\ldots\ldots\ldots\ldots+n\ldots.an..AP,\Rightarrow S_n=\frac{n}{2}(n+1)$$

Find an expression for $\sum_{1}^{n} n^2$ =Sn for the series whose Un is n^2 There are several ways to do this .

(1) is to use the identity n^3 - $(n-1)^3$ = $3n^2$- 3n+ 1 . Then let n= 1,2,3, etc . Add up the results and Simplify to find Sn

$$n^3-(n-1)^3=3n^2-3n+1.$$
$$1^3-0=3(1)^2-3(1)^2+1$$
$$2^3-1^3=3(2)^2-3(2)+1$$
$$3^3-2^3=3(3)^2-3(3)+1$$
.

.

$$n^3-(n-1)^3=3n^2-3n+1$$
$$\Rightarrow n^3=3\sum_{1}^{n}n^2-3\sum_{1}^{n}n+n(1)\Rightarrow n^3+3\sum_{1}^{n}n-n=3\sum_{1}^{n}n^2$$
$$\Rightarrow 3\sum_{1}^{n}n^2=n^3+3\frac{n}{2}(n+1)+n\Rightarrow\sum_{1}^{n}n^2=\frac{n}{6}(2n+1)(n+1)$$

Method(2) Show by Induction That $\sum_{1}^{n} n^2 = 1 + 2^2 + 3^2 + 4^2 + \ldots . n^2 . = \frac{n}{6}(2n+1)(n+1).$

(1)Show true for n = 1 . ie Show S_1 **= 1 = 1/6{2(1) +1)(1+1)}=6/6=1 true.**
(2)Assume true for n=k ie that S_k **= k/6{(2k+1)(k+1)} .**
(3)Prove true for n =(k+1) ie S_{k+1} **= (k+1)/6{[2(k+1)+1](k+1) }=(k+1)/6{(2k+3)(k+1)}**
Proof $S_{k+1} = S_k + U_{k+1} = \frac{k}{6}(2k+1)(k+1) + (k+1)^2 = k+1\{\frac{k}{6}(2k+1) + (k+1)\} = \frac{k+1}{6}(2k^2 + 7k + 6) = \frac{k+1}{6}(2k+3)(k+2)$

this is S_{k+1} **Therefore the statement is true for n = k+1 so the statement is true for n =1 and true for n=k+1 therefore it's true for all** $n \in N$

Show by Induction that $\sum_{1}^{n} n^3 = \left(\frac{n}{2}(n+1)\right)^2$

We want to show that Sn for the Series $\sum_{1}^{n} n^3$ **={n/2(n+1)}**^{2} **.**

(1) Show true for n=1 . 1 = {1/2(1+ 1)}^{2}**true**

(2) Prove true for n = k . ie $S_k = \left(\frac{k}{2}(k+1)\right)^2$ **(3) Prove true for n = (k+1) ie**

$S_{k+1} = \left(\frac{(k+1)(k+2)}{2}\right)^2$ **We use the same technique as above ie write** $S_{k+1} = S_k + U_{k+1}$

This gives $\left(\frac{k}{2}(k+1)\right)^2 + (k+1)^3 = (k+1)^2\{\left(\frac{k}{2}\right)^2 + (k+1)\} = (k+1)^2\left(\frac{k+4k+4}{4}\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2$

Therefore the statement is true for n= k+1 .
Using the $\sum n$ **type Series . Example 1**

$$.\lim_{n\to\infty} \frac{1}{n}\sqrt{1+2+3+4+\ldots\ldots\ldots n..} == \lim_{n\to\infty}\frac{1}{n}\sqrt{\frac{n(n+1)}{2}} = \lim_{n\to\infty}\sqrt{\frac{n^2+2n}{2n^2}} = \lim_{n\to\infty}\sqrt{\frac{1+\frac{2}{n}}{2}} = \frac{1}{\sqrt{2}}$$

Example 2 : Find an expression for $\sum_{1}^{n}(n+1)(2n+3)$

$$\sum_{1}^{n}(2n+1)(n+3) = \sum_{1}^{n}2n^2 + 7n + 3 = 2\sum_{1}^{n}n^2 + 7\sum_{1}^{n}n + 3n = 2\left(\frac{n(2n+1)(n+1)}{6}\right) + 7\left(\frac{n(n+1)}{2}\right) + 3n = n\left(\frac{2n^2+24n+40}{6}\right)$$

Example 3 : In a Geometric Series $U_1 = a, U_2 = ar, U_3 = ar^2 . U_n = ar^{(n-1)}$ **Find an expression for** $U_1.U_2.U_3.U_4\ldots\ldots Un$

$$U_1.U_2.U_3.U_4\ldots.Un = a.ar.ar^2.ar^3.ar^4.ar^5\ldots..ar^{n-1}. = a^n\{r^{1+2+3+\ldots..n-1}\} = a^n\{r^{\frac{(n-1)(n)}{2}}\}$$

More Series :

If you cannot remember the methods for finding Expressions for $\sum_1^n n, \sum_1^n n^2, \sum_1^n n^3$.

Another way to find an expression for the above is to use a method called Newton's Difference formula .The method involves subtracting consecative terms until you get a line of zero's .Then using the Binomial coeffecients to find the sum of the series .Find an expression for

$\sum_1^n n^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots\ldots\ldots n^2$

$= 1 + 4 + 9 + 16 + 25$

$= 3\ldots5\ldots7\ldots9..$

$= 2\ldots2\ldots.2\ldots.2\ldots$

$= 0\ldots0\ldots0\ldots..0$

$$\sum_1^n n^2 = 1c_1^n + 3^n c_2 + 2^n c_3 = 1(n) + 3\frac{n(n-1)}{2!} + 2\frac{n(n-1)(n-2)}{3!}.$$

$$= \frac{n}{6}\{6 + 9n - 9 + 2n^2 - 6n + 4\} = \frac{n}{6}\{2n^2 + 3n + 1\} = \frac{n(2n+1)(n+1)}{6}$$

The method works for all the Sigma series .

Series The Option :

Convergence A series is said to converge if the $\lim_{n \to \infty} S_n = k$ ie the limit of the Sum exists .

The problem with this definition is that it depends on finding the S_n . Another definition which requires only the U_n is called the Ratio Test This states that a Series will converge if

$\lim_{n \to \infty} \frac{U_{n+1}}{Un} < 1$ and the Series will diverge if $\lim_{n \to \infty} \frac{U_{n+1}}{Un} > 1, \ldots$ If $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = 1$ the test is inconclusive.

Example 1 : examine for convergence the series $\sum_1^\infty \frac{x^n}{n}$,

$$U_{n+1} = \frac{x^{n+1}}{n+1}, U_n = \frac{x^n}{n} .. \lim \frac{u_{n+1}}{u_n} = \lim_{\infty} \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} = \lim_{n\to\infty} \frac{x^{n+1}}{n+1} \times \frac{n}{x^n} = \lim_{n\to\infty} x \frac{n}{n+1} = x$$

The Series converges if x < 1 . The series diverges if x >1 .

Example 2

Test the Series $\sum_1^\infty \frac{x^n}{n!}$ for convergence

$$U_n = \frac{x^n}{n!}, U_{n+1} = \frac{x^{n+1}}{n+1!} :.. \lim_{n\to\infty} \frac{u_{n+1}}{U_n} = \lim_{n\to\infty} . \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{X^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0.$$

The Series converges

Special type of Series the MacLaurin Series .

This is a method which enables us to write functions such as e^x, Sinx, $Log_e x$ as a Series . The theory goes as follows If you are Given a polynomial f(x) such that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \ldots\ldots\ldots a_nx^n \Rightarrow f(0) = a_0$$

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \ldots\ldots \Rightarrow f'(0) = a_1$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 \ldots \Rightarrow f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2}$$

$$f^3(x) = 6a_3 + 24a_4x \ldots\ldots \Rightarrow f^3(0) = 6a_3 \Rightarrow a_3 = \frac{f^3(0)}{6} = \frac{f^3(0)}{3!} \Rightarrow$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^3(0)}{3!} \ldots\ldots\ldots\ldots$$

This enables us to write the polynomial as a power series :
Example write e^x as a power series

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1, f'(x) = e^x \Rightarrow f'(0) = 1, f''(x) = e^x f''(0) = 1 \Rightarrow$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots\ldots\ldots \frac{x^{n+1}}{n+1}x^{n+1}$$

Example 2
Write f(x) = Sinx as a power series
f(0) = Sin (0) = 0 , f'(x) = Cos x ,f'(0) = Cos (0) = 1, f''(x) = -Sinx ,f''(0) = -Sin (0) = 0 , f'''(x) = -Cosx , f'''(0) = -1 . We can see that a pattern is beginning to emerge

$$Sinx = 0 + x - \frac{0}{2}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 \Rightarrow Sinx = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$$

Example 3 :
Write the first four terms of the Maclaurin series for $f(x) = \sqrt{1+x}$

$$f(x) = \sqrt{1+x} \Rightarrow f(0) = 1, f'(x) = 1/2\{1+x\}^{\frac{-1}{2}} \Rightarrow f'(0) = \frac{1}{2}, f''(x) = -1/4\{1+x\}^{\frac{-3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$$

$$f^3(x) = 3/8\{1+x\}^{\frac{-5}{2}} \Rightarrow f^3(0) = 3/8. \Rightarrow$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2} + \frac{3}{8}\frac{x^3}{3!} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

Since the expansion converges for -1 < x < 1 , use the expansion to evaluate $\sqrt{10}$ correct to 1 place of decimals . We must first write $\sqrt{10}$ in the form $\sqrt{1+x}$ as follows

$\sqrt{10} = \sqrt{9+1} = 3\sqrt{1+\frac{1}{9}}$ **we now replace the 'x' in the expansion by 1/9 . This gives**

3{1+1/2(1/9) - 1/8(1/81) +1/16{1/'729) } = 3(1 .05409808 } = 3.16229424 = 3.2 to one place of decimals .

Maclauren Series for $\frac{1}{1+x} .. \frac{1}{1+x^2} . \frac{1}{\sqrt{1+x}} .,, \tan^{-1} x.$

The problem with finding a Maclaurin Series for this type of function is that after a very short time the differentiation becomes very complicated : There are several ways around this

Example 1:

$$f(x)=\frac{1}{1+x}\Rightarrow f(0)=1,..f'(x)=-1(1+x)^{-2}\Rightarrow f'(0)=-1,..f''(x)=2(1+x)^{-3}\Rightarrow f''(0)=2, f^{3}(x)=-6(1+x)^{-4}\Rightarrow f^{3}(0)=-6$$

this gives us the MacLauren Series

for $\frac{1}{1+x}=1-x+\frac{2x^2}{2!}-\frac{6x^3}{3!}+\frac{24x^4}{4!}\ldots\ldots\ldots=1-x+x^2-x^3+x^4$

We can use this series to find the Maclaurin series for $\log_e(1+x)$ **as follows**

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5\ldots\ldots$$

$$\int\frac{1}{1+x}dx=\int 1-x+x^2-x^3+x^4-x^5+\ldots dx$$

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}\ldots.$$

From above the Maclaurin Series for $\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8-x^{10}\ldots\ldots$ **so a quick way to find the Maclaurin Series for** $\tan^{-1}x$ **is to do the following**

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+x^8.\int\frac{1}{1+x^2}dx=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}=\tan^{-1}x$$

We can use the series for Tan^{-1}X to find a value for Π

$\tan^{-1}X+\tan^{-1}Y=\tan^{-1}\{\frac{X+Y}{1-XY}\}$ **We use this and the fact that**

$\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\frac{1}{3}}\right)=\tan^{-1}\frac{\frac{5}{6}}{\frac{5}{6}}=\tan^{-1}1=\frac{\Pi}{4}$ **So replace x in the series above first by 1/2 then by 1/3 add the results this gives** $\frac{\Pi}{4}$ **now multiply your answer by 4 .**

$$\tan^{-1}\frac{1}{2}=\frac{1}{2}-\frac{\frac{1}{2}^{3}}{3}+\frac{\frac{1}{2}^{5}}{5}:..=\frac{223}{480}=0.464583$$

$$\tan^{-1}\frac{1}{3}=\frac{1}{3}-\frac{\frac{1}{3}^{3}}{3}+\frac{\frac{1}{3}^{5}}{5}=0.3215$$

$$\Rightarrow\frac{\Pi}{4}=0.786083\Rightarrow\Pi=3.14433$$

this is only an approximation for Π **a better result could be found using more terms of the series .**