

**Linear Transformations :**

Transformation This is an operation in the Coordinate plane or vector space

which maps  $(x, y) \rightarrow (x_1, y_1)$ . or,  $(x\mathbf{i} + y\mathbf{j}) \rightarrow (x_1\mathbf{i} + y_1\mathbf{j})$

Examples of transformations are (1) Reflections in the X axis or a Central Symmetry in (0,0) ie  $S_x(x, y) \rightarrow (x, -y)$ .  $S_0(x, y) \rightarrow (-x, -y)$

Other transformations can be written as follows  $\begin{matrix} x_1 = ax + by \\ y_1 = cx + dy \end{matrix}$  which can be

expressed as  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is called the **transformation matrix** of the transformation

The Course requires that we find the image of points by a given transformation If f is a transformation such that

$x_1 = 3x + y$  ie  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$  if a(0,0) b (2,1) c(3,4) Find f(a),f(b),f(c) The simplest

way is to **Multiply the transformation matrix by each of the points** ie .....a.b.c...f(a).f(b).f(c)

$\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 13 \\ 10 \end{pmatrix}$  We are often asked to write x and y in terms of  $(x_1, y_1)$ .

To find x and y we know  $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$  premultiply both sides of the equation by

the inverse of the transformation matrix . The inverse of  $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = 1/1 \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$  this

gives  $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow x = x_1 - y_1 \Rightarrow y = -2x_1 + 3y_1$  . We can use this relationship to

find the image of a Line or a Line segment by the transformation just replace x and y by their images in  $(x_1, y_1)$

**Ex : find the image of the line  $3x + 4y + 5 = 0$  by f the result is  $3(x_1 - y_1) + 4(-2x_1 + 3y_1) + 5 = 0 = -5x_1 + 9y_1 + 5 = 0$ . It is a requirement that we show the inverse to be true ie  $-5(3x+y) + 9(2x+y) + 5 = 0 = 3x + 4y + 5 = 0$  .**

**Example if  $H : ax + by + c = 0$  and  $K : ax + by + d = 0$  are 2 parallel lines Show f(H) and f(K) are parallel**

$f(H) = a(x_1 - y_1) + b(-2x_1 + 3y_1) + c = 0 \Rightarrow x_1(a - 2b) + y_1(-a + 3b) + c = 0$   
 $f(K) = a(x_1 - y_1) + b(-2x_1 + 3y_1) + d = 0 \Rightarrow x_1(a - 2b) + y_1(-a + 3b) + d = 0$  **both have**

slope  $\frac{-a + 2b}{-a + 3b}$  therefore f(H) is parallel to f(K)

**Example if  $L : px + qy + r = 0$  and  $N: qx - py + t = 0$  are 2 perpendicular lines investigate if f(L) is perpendicular to f(N)**

$f(L) = p(x_1 - y_1) + q(-2x_1 + 3y_1) + r = 0 \Rightarrow x_1(p - 2q) + y_1(-p + 3q) + r = 0$   
 $f(N) = q(x_1 - y_1) - p(-2x_1 + 3y_1) + t = 0 \Rightarrow x_1(q - 2p) + y_1(-q - 3p) + t = 0$

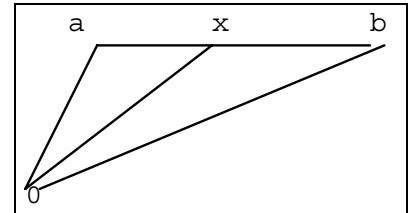
The slope of  $f(L)$  is  $-p + 2q$  the slope of  $f(N)$  is  $-q + 2p$  the product of the slopes is not equal to  $-1$

$-p + 3q$   $-q - 3p$   $f(L)$  is not perpendicular to  $f(N)$

To write a Line segment in parametric form we use an idea from vectors

If  $x$  is any point on a line segment  $[ab]$  where  $o$  is the origin then

$$\vec{a} + t\vec{ab} = \vec{x} \Rightarrow \vec{a} + t(\vec{b} - \vec{a}) = \vec{x} \Rightarrow t\vec{b} + (1-t)\vec{a}$$



This enables to write any line segment  $[ab]$  in parametric form if  $t = 0$   $x = a$ , if  $t = 1$   $x = b$  if  $x \in [ab] \Rightarrow 0 \leq t \leq 1$  To find the image of a line segment by

a transformation just multiply the transformation matrix by the Line segment written in parametric form .

NOTE if  $a = (x_1, y_1), b = (x_2, y_2) \Rightarrow x = t(x_2) + (1-t)x_1, y = t(y_2) + (1-t)y_1$

EXAMPLE  $a = (1, 2), b = (4, 1)$  write  $[ab]$  in parametric form . We know if  $p$  is a point on  $[ab]$  then  $p = tb + (1-t)a$  gives  $x = 4t + (1-t)1 = x = 3t + 1, y = 1t + (1-t)2 = y = 2 - t$  So the image of  $[ab]$  by  $f$  is

$$\begin{pmatrix} 3 & \dots & 1 \\ 2 & \dots & 1 \end{pmatrix} \begin{pmatrix} 3t+1 \\ 2-t \end{pmatrix} = \begin{pmatrix} 8t+5 \\ 5t+4 \end{pmatrix} \quad \mathbf{x} =$$

$$8t + 5, y = 5t + 4$$

## Properties Of linear Transformations

If  $f$  is a Linear transformation then

- (1)  $f(0) = 0$  . The origin is it's own image .
- (2) If  $a$  and  $b$  are two points then  $f(a+b) = f(a) + f(b)$  .
- (3) The image of a line is a line , the image of a line segment is a line segment.
- (4) the image of a pair of parallel lines is a pair of parallel lines .
- (5) If  $L$  and  $K$  are 2 lines and  $L \cap K = p \Rightarrow f(L) \cap f(K) = f(p)$ .
- (6) The Image of a parallelogram is a parallelogram .
- (7) The images of two perpendicular lines are not necessarily perpendicular .
- (8) If  $A$  is the area of a triangle  $abc$ , the area of  $f(abc)$  is  $kA$  where  $k$  is the determinant of the transformation matrix .
- (9) When asked to find the image of a line in the form  $ax+by+c = 0$  by a transformation not only must you find it's image but you must also transform it back to verify that all of the original line is mapped onto the image . If the line is in parametric form there is no need for the inverse transformation .

## Properties of 2x2 Matrices

(1) If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} x & z \\ y & w \end{bmatrix} \Rightarrow A+B = \begin{bmatrix} a+x & b+z \\ c+y & d+w \end{bmatrix}, AB = \begin{bmatrix} ax+by & az+bw \\ cx+dy & cz+dw \end{bmatrix}$

(2)  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  The trace of  $A$  is  $a + d$ , (3) The determinant of  $A$  is  $ad - bc$ .

(4) The Characteristic equation of  $A$  is the

Quadratic  $x^2 - (\text{trace } A)x + \text{determinant} = 0$ . ie..  $x^2 - (a+d)x + ad - bc = 0$ .

(5) the Inverse of Matrix A is  $= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1} \dots A.A^{-1} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(6) Note If A and B are two Matrices Show

$(A^{-1}BA)^n = A^{-1}B^nA \dots$

$(A^{-1}BA)^n = A^{-1}BAA^{-1}BAA^{-1}BA \dots = A^{-1}BBBBBBB \dots A = A^{-1}B^nA \dots (A^{-1}A) = I$

(7) A diagonal Matrix is a Matrix of the form  $K = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$  note  $K^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$