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Notes on Induction

Induction is a process which uses Three steps to show that a given statement is true ex we can show by induction that $\sum \mathbf{n}^2 = \frac{\mathbf{n(2n+1)(n+1)}}{\mathbf{6}}, \mathbf{n} \in \mathbf{N_0}$,, or that $\mathbf{7^{2n+1}7^{2n+1} + 1, n \in N \geq 1}$ is divisible by 8, or $\mathbf{(1+x)^n \geq 1 + nx, x > -1, n \in N_0}$. Note all the statements refer to n as a positive integer , Induction is a process which applies to statements involving **positive integers** only !.

The process works as follows (1) We show that the statement is true for **n = 1 ,** (2) We assume that it is true for **n= k** ,

(3) We prove the statment is true for **n = k+1** . The idea being that if the statement is true for n = 1 and is true for n = k +1 , if true for n = k . Then if n = 1 it's true for 1+1, if true for 2 then it's true for 2 + 1 , etc for all positive integers .

One way in which Induction is described is that of an Infinite ladder and the rungs of the ladder are so spaced that if you are on any rung you know you can get to the next rung, which means if you can get to the first rung you can climb the ladder as high as you want .

Mathematical Induction is used in the following situations
(1) To show that a given expression in n represents the sum of the first n terms of a particular Series .
(2) To show that certain Inequalities are true .
(3)To Show that a given expression is divisible by a given number .
(4) To show DeMoivres theorem is true, (5) To show The Binomial Theorem is true,
(6) To show that $\mathbf{\frac{d(x^n)}{dx} = nx^{n-1}}$.

Sums of Series

Example 1 Prove $\sum_{1}^{n} \mathbf{r^2 = 1^2 + 2^2 + 3^2 + 4^2 ....n^2 .= \frac{n(2n+1)(n+1)}{6} = S_n}$.

(1)For n = 1. We show $\mathbf{S_1 = 1 = \frac{1.3.2}{6} = 1.}$ the statement is true .

(2)We now assume that the statement is true for n = k , ie $\mathbf{S_k = \frac{k(2k+1)(k+1)}{6}}$

(3) We now prove that the statement is true for n = k+1. ie $\mathbf{S_{k+1} = \frac{(k+1)(2k+3)(k+2)}{6}}$.

The way to prove this is follows $\mathbf{S_{k+1} = S_k + U_{k+1}}$ =

$$\mathbf{\frac{k(2k+1)(k+1)}{6} + (k+1)^2 .. = \frac{k(2k+1)(k+1) + 6(k+1)^2}{6}}$$

$$\mathbf{= \frac{(k+1)\{k(2k+1) + 6(k+1)\}}{6} = \frac{(k+1)\{2k^2 + 7k + 6)\}}{6} = \frac{(k+1)(2k+3)(k+2)}{6} = S_{k+1}}$$

We have shown the statement to be true for n = 1, and if true for n = k , it's true for n = k + 1 ,

This Method of using $\mathbf{S_{k+1} = S_k + U_{k+1}}$ will always work for sums of series .

Factorisation Results ("is divisible by")

The srandard method here is to write the expression for k + 1 in terms of the expression in terms of k and a remainder , the remainder must be divisible by the given number .

Ex (2) Prove $5^n - 3^n$ is divisible by 2 .If n = 1 we get 5 - 3 = 2 which is divisible by 2 . Assume the statement is true for n = k , ie $5^k - 3^k$ is divisible by 2 .

Now prove the statement is true for n = k+ 1 . ie $5^{k+1} - 3^{k+1}$ is divisible by 2 .$5^{k+1} - 3^{k+1} = 55^k - 33^k = 3(5^k - 3^k) + 25^k$

We know$5^k - 3^k$ is divisible by 2 and 25^k is divisible by 2 therefore the statement is true for n = k+1

Inequalities

Prove $2^n > n^2,..n > 4$

(1)Show the statement is true for n = 5 . 32 .> 25 True we use 5 as n > 4 .

(2)Assume the statement is true for n = k , $2^k > k^2.$

(3) Prove true for n = k + 1 .

$$2^{k+1} > (k+1)^2 \Rightarrow 2(2^k) > k^2 + 2k + 1,..\Rightarrow$$

$$2^k + 2^k > k + 2k + 1,but.2^k > k^2..and.2^k > 2k + 1..for..k > 4.$$

Note in step (3) the RHS is a product wheras the LHS is a Sum it is important to **change both sides** into a SUM or a product .

Example 2:

Prove $2^n \geq 1 + n, \forall.n \in N$

(1)Prove true for n = 1 $2 \geq 1 + 1..\Rightarrow 2 \geq 2..true!$

(2)Assume true for n = k . $2^k \geq k + 1$. Now prove true for n = k+1

$$2^{k+1} \geq k + 1 + 1 \Rightarrow 2(2^k) \geq k + 1 + 1 \Rightarrow 2^k + 2^k \geq k + 1 + 1$$

$$but..2^k \geq k + 1..and.2^k \geq 1,..for..k \in N.$$

Example 3.

Based on the binomial theorem

Prove $(1+x)^n \geq 1 + nx,..x > 0.$

(1)Prove true for n = 1. $(1+x) \geq 1 + x..true!$

(2)Assume true for n = k . ie $(1+x)^k \geq 1 + nx$. now prove true for n = k+1 .

(3)
$$(1+x)^{k+1} \geq 1 + (k+1)x \Rightarrow (1+x)^k(1+x) \geq 1 + kx + x \Rightarrow$$
$$(1+x)^k + x(1+x)^k \geq (1+kx) + x..but(1+x)^k \geq (1+kx)..and..x(1+x)^k \geq x,..\forall x > 0$$

therefore the statement is true for n = k + 1 .

Some interesting Examples :

Show $x^n - y^n$ is divisible by **x - y** . (1) Show true for n = 1 , ie (x - y) is divisible by (x - y)

(2) Assume true for n = k . ie $x^n - y^n$ is divisible by **x - y** .

(3) Prove true for n = k + 1 . is

$$x^{k+1} - y^{k+1} = x^{k+1} - x^k y + x^k y - y^{k+1} = x^k(x - y) + y(x^k - y^k)$$

To prove the statement true we add and subtract $x^k y$ we can see from the result that **(x - y)** is a factor of both parts .

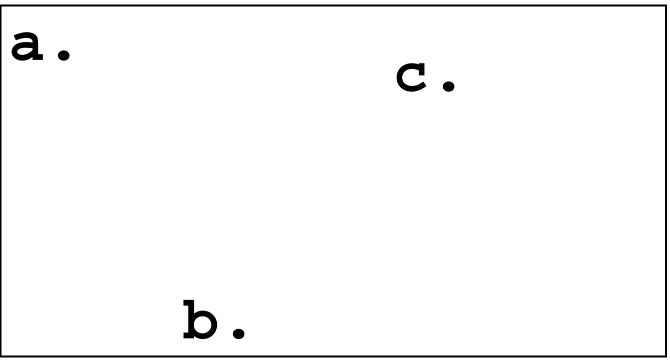
Prove that (n)(n-1) couples (excluding identity couples) can be formed from n points .

(1)When n = 2 we get two couples =2(2-1) = 2 true for n = 2 .

(2)Assume true for **n = k ie we get k(k-1) couples from k points**

(3) Prove true for **n = k + 1 . ie we should get (k+1)k couples from k + 1 points** .

For **n = k we get k(k-1**) couples when n = k + 1 (**we get an extra 2k couples so the total number of couples from k + 1 points is k(k - 1) + 2k = k(k + 1) the statement is true for n = k + 1 .



** if you have k points and add an extra point yuo will get an extra k couples

When we have two points a and b we have two couples (a,b) and (b,a) When we add the point c we get 4 more couples ie (a,c),(c,a),(c,b)(b,c) that is 2 couples for each point that is already there .